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LAMINAR FREE-CONVECTIVE HEAT TRANSFER FROM THE OUTER SURFACE OF A VERTICAL CYLINDER WITH UNIFORM SURFACE HEAT FLUX

A THESIS

Presented to

the Faculty of the Graduate Division

by

Leonard Hugh Caveny

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Mechanical Engineering

Georgia Institute of Technology

July, 1960

LAMINAR FREE-CONVECTIVE HEAT TRANSFER

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Approved:

Thesis Advisor

Date Approved by Chairman: 8/29/60

ACKNOWLEDGEMENTS

In particular, I would like to express my appreciation to Dr. C. W. Gorton for suggesting this topic, for acting as my advisor, and for his encouragement throughout the year. The courses of Dr. W. B. Harrison and Dr. C. W. Gorton provided the background which enabled me to approach this topic using current techniques. I appreciated the comments and additions of Dr. W. B. Harrison and Dr. J. D. Fleming, who read this thesis. The willing cooperation I received from everyone was most gratifying.

I would like to acknowledge the perseverance of my wife who somehow found time between keeping the children quiet and presenting me with my third daughter to type the rough draft.

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SUMMARY

The heat transfer results for laminar free convection on a right vertical circular cylinder with uniform surface heat flux were determined analytically. These results apply to a vertical cylinder of small enough radius that the effect of curvature on the heat transfer must be considered.

The boundary-layer equations and boundary conditions were expressed in cylindrical coordinates. An exact solution of the boundary-layer equations could not be found and the approximate method of Karman and Pohlhausen was used. This method involves writing the velocity and temperature profiles as polynomials whose coefficients are determined from the boundary conditions, and integrating the boundary-layer equations over the boundary-layer thickness. Introducing the velocity and temperature profiles into the integrated form of the boundary-layer equations and applying the boundary conditions resulted in a pair of ordinary coupled non-linear differential equations. These equations were solved by a laborious series technique for the Prandtl number range of 0.01 to 1,000. For the Prandtl number greater than 100 a closed form solution, that did not depend on the convergence of a series, was obtained.

The heat transfer results were presented as curves for values of the Prandtl number equal to 0, 0.1, 1, 10 and for Prandtl number greater than 100. Using these curves, if the properties of the fluid and the surface heat flux are known, the following can be found:

1. Local Nusselt number
2. Average Nusselt number
3. Surface temperature
4. Boundary-layer thickness

The results demonstrate readily how the laminar free-convection heat transfer results for a vertical cylinder with uniform surface heat flux differ from a vertical flat plate with uniform surface heat flux due to the effect of curvature. When items 1 to 4 (as stated above) are compared to a vertical flat plate, under the condition that the surface heat flux and vertical height are the same, there was shown to be up to an 11 per cent difference for the range of the solution.

NOMENCLATURE

a - radius of cylinder, ft

A - area, ft^2

A_o, A_1, A_2 - coefficients of powers of x expressing u_o, u_1, u_2

B_o, B_1, B_2 - coefficients of powers of x expressing δ_o, δ_1, and δ_2

D - diameter of cylinder, ft

g - acceleration due to gravity, ft/sec^2

h_x - local heat transfer coefficient, Btu/sec ft^2 °F

$\bar{h}$ - average heat transfer coefficient, $\frac{q}{\bar{\theta}_w}$, Btu/sec ft^2 °F

$G_{r_x}^*$ - modified Grashof number, $\left(\frac{g\beta q x^4}{K\nu^2}\right)$

G_{r_x} - Grashof number, $\left(\frac{gD^3\beta\theta}{\nu^2}\right)$

K - thermal conductivity, Btu/sec ft °F

n - index for terms in series, dimensionless

L - vertical length, ft

Nu_x - Nusselt number based on x $\left(\frac{h_x x}{K}\right)$

$\overline{Nu}$ - average Nusselt number $\left(\frac{\bar{h}x}{K}\right)$

Pr - Prandtl number

q - local heat-transfer rate per unit area, Btu/sec ft^2

Q - heat transfer rate, Btu/sec

r - co-ordinate measuring radial distance from cylinder axis, ft

t - static temperature, °F

t_e - ambient temperature, °F

t_w - wall temperature, °F

u - velocity in x direction, ft/sec

v - velocity in r direction, ft/sec

x - co-ordinate measuring longitudinal distance along cylinder, ft

y - distance from surface of cylinder, r - a, ft

α - thermal diffusivity, ft^2/sec

$\alpha_1, \alpha_2, \ldots$ - coefficient of x expressing $A_1, A_2, \ldots$

β - coefficient of thermal expansion, °F^{-1}

$\delta_1, \delta_2, \ldots$ - coefficient of x expressing $B_1, B_2, \ldots$

θ - temperature excess, $t - t_e$, °F

θ_w - temperature excess of wall, $t_w - t_e$, °F

$\bar{\theta}_w$ - average temperature excess of wall, °F

ν - kinematic viscosity, ft^2/sec

δ - boundary layer thickness, ft

ϕ - dimensionless independent variable

$$\left\{ 2\, B_o \frac{x^{\frac{1}{5}}}{D} \right\}$$

ϕ_L - dimensionless independent variable

$$\left\{ 2\ B_o\ \frac{L^{\frac{1}{5}}}{D} \right\}$$

Subscripts

cyl - cylinder

fp - flat plate

w - wall

Dimensionless Terms

$$\tilde{r} = \frac{r}{a}$$

$$\tilde{x} = \frac{x}{a}$$

$$\tilde{v} = \frac{av}{\nu}$$

$$\tilde{u} = \frac{au}{\nu}$$

$$\tilde{\delta} = \frac{\delta}{a}$$

CHAPTER I

INTRODUCTION

If an object is placed in a fluid that is at a higher or a lower temperature than it is, free-convection heat transfer occurs. The resulting heat transfer causes a temperature difference within the fluid, which results in a change in density of the fluid layers in the vicinity of the surface; and consequently, due to buoyancy, there is a fluid flow. As in other fluid flows, free-convection can be either laminar or turbulent depending on the fluid properties, the body force, distance from leading edge, etc. Although free convection is commonly considered as an effect of the gravitational field of the earth, many recent developments have caused interest in free convection where gravity is not the only consideration, and effects such as those caused by electric, magnetic, and centrifugal forces must also be considered.

The mechanism of free-convection heat transfer has been the subject of considerable experimental and analytical work. The solutions for the free-convection boundary-layer equations for the vertical flat plate have been generally established, but up to 1955 very few analytical results had been published for the vertical cylinder. Some experimental correlations had been published but in general they were quite limited. As free-convection solutions for the flat plate were refined it became increasingly more important to know to what degree the curvature of a cylinder would affect these results.

In 1955 Sparrow and Gregg [1]* were able to obtain an approximate solution for the boundary-layer equations for a vertical cylinder with uniform surface temperature. A disadvantage of this solution was that it had to be carried out for each value of Prandtl number. LeFevre and Ede [2] were able to obtain a solution for the same case using the Karman-Pohlhausen integral method which could be evaluated directly for any value of the Prandtl number.

Hama and Recesso [3] were concerned with the case in which the boundary-layer thickness was greater than the radius of the cylinder and used a modified form of the Karman-Pohlhausen integral method. The solution obtained was for Prandtl numbers greater than 0.72.

The only available exact solution of the boundary-layer equations for a cylinder was obtained by Millsaps and Pohlhausen [4] for the case of linear surface temperature variation.

Sparrow [5] obtained a solution for laminar free convection on a vertical flat plate with uniform surface heat flux using the Karman-Pohlhausen integral method. Then Sparrow and Gregg [6] obtained an exact solution of the boundary-layer equations for the same case, which verified the previous results obtained by using the Karman-Pohlhausen integral method.

The object of the present work is to obtain an analytical expression for the heat transfer parameters associated with laminar free convection on a vertical cylinder with uniform surface heat flux.

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*Numbers in brackets refer to the Bibliography.

CHAPTER II

MATHEMATICAL FORMULATION

The geometric shape under consideration is a right circular vertical cylinder with small enough radius that curvature effects must be considered. Following Sparrow and Gregg [1] the two physical models that come within the scope of this analysis are presented in Figure 1.

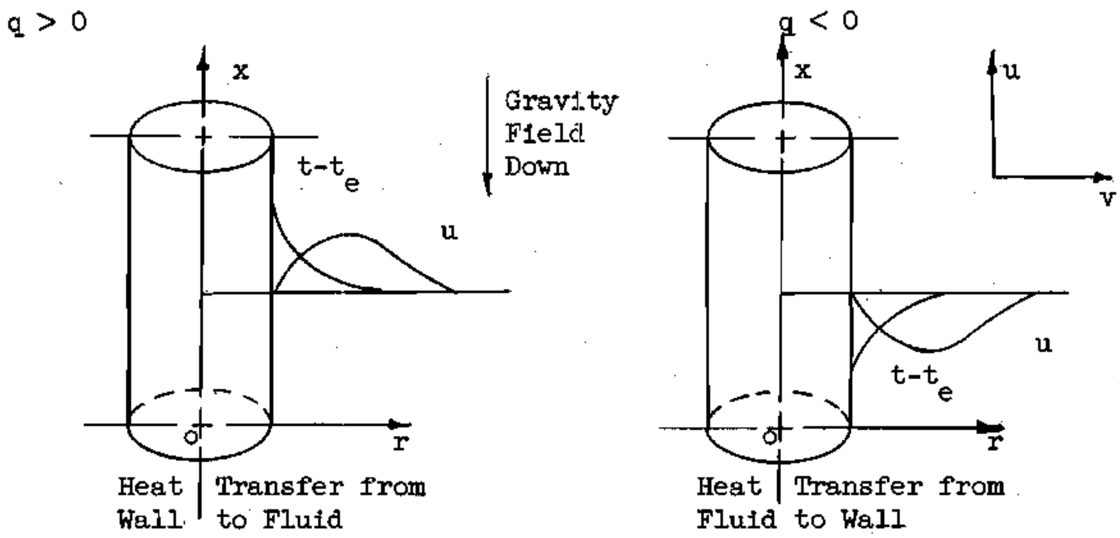

Figure 1. Diagram of Heated Surface Showing Coordinate System and Notations

In the left-hand sketch, the fluid in the neighborhood of the wall has a higher temperature and a lower density than the fluid further from the wall (this applies to ordinary fluids which decrease in density as temperature increases). Thus, due to buoyancy, the fluid in the neighborhood of the wall will flow up. The region where this upward flow occurs is called the velocity boundary layer for free convection. The thermal boundary layer is defined as the region where the temperature deviates from the ambient temperature. In practice the thermal boundary-layer thickness and the velocity boundary-layer thickness are defined somewhat arbitrarily since the exact solution of the boundary equations results in velocity and temperature profiles that are asymptotic. Thus the thermal boundary layer thickness can be defined as a point where the temperature of the fluid is arbitrarily close to the ambient temperature, and in a similar manner the velocity boundary-layer thickness is defined at a point where the velocity away from the wall is arbitrarily close to zero. Both of these boundary layers will be assumed to have zero thickness at the leading edge.

In the right-hand sketch the temperature in the neighborhood of the wall is lower, and the resulting flow will be down. If the coordinate system of Figure 1 is used, the heat transfer results can be applied equally well to either case.

The steady state partial differential equations for laminar free convection in a boundary layer on a vertical cylinder as stated by Sparrow and Gregg [1] are

Conservation of Mass

$$\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial r} = 0 \tag{1}$$

Conservation of Momentum

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = g\beta(t-t_e) + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \tag{2}$$

Conservation of Energy

$$u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial r} = \frac{\alpha}{v}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right) \tag{3}$$

Following the usual practice in free convection, all fluid properties will be taken as constant except density, which will be considered a variable in formulating the buoyancy term, $g\beta(t-t_e)$. Viscous dissipation and work against the gravity field will be neglected.

It would be desirable to obtain an exact solution for the above equations. According to Yang [7] the only exact solutions that have been obtained for these equations were derived by using the technique of similarity solution. The technique consists of finding a suitable transformation that will reduce the partial differential equations to a set of ordinary differential equations, and then the resulting equations with the boundary conditions can be solved numerically. The only available similarity solution for a vertical cylinder was published by Millsaps and Pohlhausen [4]. An unsuccessful attempt was made to find a "suitable transformation" for the vertical cylinder with uniform surface heat flux. Thus it will be necessary to make certain simplifying assumptions in order to employ another method of solution.

Following Eckert and Drake [8] it will be assumed that the boundary-layer thickness for velocity and temperature have the same value. This assumption has been justified several times for laminar free convection on vertical surfaces. In particular, Eckert and Drake [9] showed that for the case of a vertical plate with uniform surface temperature the agreement between solutions by H. B. Squire, using this assumption, and an exact solution by S. Ostrach was good in the Prandtl number range of 0.01 to 1,000. Sparrow and Gregg [1] demonstrated that results obtained by using this assumption were in good agreement with their exact solution for a vertical flat plate with uniform surface heat flux. LeFevre and Ede [2] made this assumption which led to results that were in good agreement with the more exact solution of Sparrow and Gregg [1] and experimental data for the vertical cylinder with uniform surface temperature.

Following LeFevre and Ede [2] equations (2) and (3) when integrated across the boundary-layer thickness, take the following form:

Conservation of Momentum

$$\frac{d}{dx}\int_{a}^{a+\delta} u^2 r dr = g\beta \int_{a}^{a+\delta} \theta(x,r) r dr - a\nu \left.\frac{du}{dr}\right)_{r=a} \tag{2a}$$

Conservation of Energy

$$\frac{d}{dx}\int_{a}^{a+\delta} \theta(x,r) u r dr = - a\alpha \left.\frac{\partial \theta}{\partial r}\right)_{r=a} \tag{3a}$$

The corresponding boundary conditions are

1. $\frac{\partial \theta(x,a)}{\partial r} = - \frac{q}{K}$

2. $\theta(x,a) = \theta_w(x)$

3. $u(x,a) = 0$

4. $\theta(x,\delta) = 0$

5. $u(x,\delta) = 0$

6. $\delta(0) = 0$

7. $u(0,r) = 0$

The more convenient dimensionless form of equations (2a) and (3a) as developed in Appendix A is:

Conservation of Momentum

$$\frac{d}{d\tilde{x}}\int_0^{\tilde{\delta}} \tilde{u}^2(\tilde{y}+1)d\tilde{y} = \frac{g\beta a^3}{\nu^2}\int_0^{\tilde{\delta}} \theta(x,r)(1+\tilde{y})d\tilde{y} - \left.\frac{\partial\tilde{u}}{\partial\tilde{y}}\right)_{\tilde{y}=0} \tag{2b}$$

Conservation of Energy

$$\frac{d}{d\tilde{x}}\int_0^{\tilde{\delta}} \theta(x,r)\tilde{u}(1+\tilde{y})d\tilde{y} = -\frac{\alpha}{\nu}\left.\frac{\partial\theta(x,r)}{\partial\tilde{y}}\right)_{\tilde{y}=0} \tag{3b}$$

The Karman-Pohlhausen integral method, which involves writing temperature and velocity profiles as polynomials in y whose coefficients are functions of x, will be used. By using this method, the differential equations of boundary layer flow are satisfied only in the average and over the boundary-layer thickness. An exact solution, if available, would satisfy the boundary conditions for every individual fluid particle. The expressions of the velocity and temperature profiles as polynomials in y have been verified by Eckert and Drake [9] from the exact solution for these profiles for the flat plate by E. Schmidt and E. Pohlhausen. The form of the coefficients are arrived at from the boundary conditions.

It was mentioned that the technique of similarity solution led to an exact solution of the boundary-layer equations. In order to use this method, similar velocity and temperature profiles must exist, i.e., a velocity profile (or temperature profile) at any position x along the laminar region has the same shape and differs only by a scale factor from any other velocity profile (or temperature profile) in the laminar region. When these conditions exist, the use of a polynomial profile would approximate the situation along the entire surface and the results obtained using such profiles have been in good agreement with the exact solutions. In particular, for laminar free convection, Sparrow and Gregg [6] showed that the results obtained using these profiles were in good agreement with their exact solution for the vertical flat plate with uniform heat flux. However, for the cases where the technique of similar solution has not been applied it has been necessary to assume similar profiles and then proceed with the Karman-Pohlhausen integral method. Experimental justification of the assumption would be necessary. This was done by LeFevre and Ede [2] for the vertical cylinder with uniform surface temperature and the results were shown to agree with the less approximate solution of Sparrow and Gregg [1], which did not make this assumption, and with the available experimental data.

From H. B. Squire as presented by Eckert [10]

$$u(x,y) = u^*(x)\frac{y}{\delta}[1-\frac{y}{\delta}]^2 \quad (4)$$

$$\theta(x,y) = \theta_w(x)(1-\frac{y}{\delta})^2 \quad (5)$$

By inspection, equation (4) can be seen to satisfy the boundary conditions three and five. Similarly, equation (5) satisfies boundary conditions two and four. Since in the statement of the problem δ and u were to be zero at the leading edge, i.e., boundary conditions six and seven, it will be necessary to define

$$u^*(0) = 0$$

By boundary condition one

$$q = \text{constant} = -K\left.\frac{\partial\theta}{\partial y}\right)_{y=0}$$

and by introducing first derivative of the temperature profile

$$q = -K\,\theta_w(x)\,2(1-\frac{y}{\delta})(-\frac{1}{\delta})_{y=0}$$

$$= \frac{2K\,\theta_w(x)}{\delta}$$

or

$$\theta_w = \frac{q\delta}{2K} \quad (6)$$

Substitution of equations (4), (5) and (6) in the integrated form of the dimensionless expression for conservation of momentum, equation (2b), results in

$$\frac{d}{d\tilde{x}}\int_{0}^{\tilde{\delta}}\left\{ u^{*}(x)\,\frac{\tilde{y}}{\tilde{\delta}}\left[1-\frac{\tilde{y}}{\tilde{\delta}}\right]^{2}\right\}^{2}(1+\tilde{y})d\tilde{y} =$$

$$\frac{g\beta a^{3}}{\nu^{2}}\int_{0}^{\delta}\theta_{w}(x)\,\left(1-\frac{\tilde{y}}{\delta}\right)^{2}\,(1+\tilde{y})d\tilde{y}$$

$$-\left.\frac{\partial\left(u^{*}(x)\,\frac{\tilde{y}}{\tilde{\delta}}\left[1-\frac{\tilde{y}}{\tilde{\delta}}\right]^{2}\right)}{\partial\tilde{y}}\right)_{\tilde{y}=0}$$

Upon performing indicated operations and simplifying the expression for conservation of momentum becomes

$$\frac{1}{840}\,\frac{d}{dx}\left\{\tilde{u}^{*}\left[8\,\tilde{\delta}+3\,\tilde{\delta}^{2}\right]\right\} \tag{2c}$$

$$=\frac{1}{24}\left(\frac{g\beta qa^{4}}{\nu^{2}K}\right)\left(4\,\tilde{\delta}^{2}+\tilde{\delta}^{3}\right)-\frac{\tilde{u}^{*}}{\tilde{\delta}}$$

In the same manner, substitution in the dimensionless expression for conversation of energy, equation (3b), results in

$$\frac{d}{d\tilde{x}} \int_{o}^{\tilde{\delta}} u^*(x) \frac{\tilde{y}}{\tilde{\delta}} \left[1 - \frac{\tilde{y}}{\tilde{\delta}}\right]^2 \theta_w(x) \left[1 - \frac{\tilde{y}}{\tilde{\delta}}\right]^2 (1 + \tilde{y}) d\tilde{y}$$

$$= - \frac{\alpha}{\nu} \frac{\partial}{\partial \tilde{y}} \left(\theta_w(x) \left[1 - \frac{\tilde{y}}{\tilde{\delta}}\right]^2 \right)_{\tilde{y}=0}$$

Upon performing indicated operations and simplifying the expression for conservation of energy becomes

$$\frac{d}{d\tilde{x}} \left\{ \theta_w(x) u^*(x) \left[\frac{\tilde{\delta}}{30} + \frac{\tilde{\delta}^2}{105} \right] \right\}$$

$$= \frac{2 \ \theta^*(x)}{Pr \ \tilde{\delta}}$$

Introducing equation (6)

$$\theta_w = \frac{q\delta}{2K} = \frac{qa\tilde{\delta}}{2K}$$

results in

$$\frac{d}{d\tilde{x}} \left\{ \frac{qa\tilde{\delta}}{2K} u^* \left[\frac{\tilde{\delta}}{30} + \frac{\tilde{\delta}^2}{105} \right] \right\}$$

$$= \frac{2 \ qa}{Pr 2K}$$

or after simplifying

$$\frac{d}{d\tilde{x}} \left\{ u^* \left[\frac{\tilde{\delta}^2}{30} + \frac{\tilde{\delta}^3}{105} \right] \right\} = \frac{2}{Pr} \qquad (3c)$$

Equations (2c) and (3c) are a pair of ordinary coupled non-linear differential equations in δ and u^*.

Equation (3c) can be integrated directly resulting in

$$u^* = \frac{1}{Pr} \left[\frac{420\ \tilde{x}}{7\tilde{\delta}^2 + 2\tilde{\delta}^3} \right] \tag{3d}$$

Now that the indicated integrations have been performed the advantages of the dimensionless form have been realized and from this point it will be more convenient to work with dimensional terms. Thus equations (2c) and (3d) become

$$\delta \frac{d}{dx} \left\{ u^* \left[8a\delta + 3\delta^2 \right] \right\} \tag{2e}$$

$$= \frac{35g\beta q}{K} \left\{ (4a + \delta)\delta^3 \right\} - 840\ a\nu u^*$$

$$u^* = \frac{\nu}{Pr} \left\{ \frac{a\ 420\ x}{7a\delta^2 + 2\delta^3} \right\} \tag{3e}$$

The foregoing analysis and formulation has led to a pair of coupled non-linear simultaneous equations, (2e) and (3e) in two unknown functions of x, δ and u^*.

CHAPTER III

SERIES SOLUTION OF BOUNDARY-LAYER EQUATIONS

Following the method that was used successfully by LeFevre and Ede [3], it will be assumed that a pair of series with the radius, a, as parameters of the form

$$\delta = \delta_o + \frac{\delta_1}{a} + \frac{\delta_2}{a^2} + \ldots \tag{7}$$

and

$$u^* = u_o + \frac{u_1}{a} + \frac{u_2}{a^2} + \ldots \tag{8}$$

can be written and that they converge. Equations (7) and (8) will be introduced into the integrated expressions of conservation of momentum and conservation of energy. Terms will be grouped according to the power of a that multiplies them.

Substituting the series into equations (2e) and (3e) gives

$$(\delta_o + \frac{\delta_1}{a} + \frac{\delta_2}{a^2} + \ldots)\frac{d}{dx}\left\{ (u_o{}^2 + \frac{2u_ou_1}{a} + \frac{2u_ou_2 + u_1{}^2}{a^2} + \ldots) \right. \tag{2f}$$

$$\left. \left[8a(\delta_o + \frac{\delta_1}{a} + \frac{\delta_2}{a^2} + \ldots) + 3(\delta_o{}^2 + \frac{2\delta_o\delta_1}{a} + \frac{2\delta_o\delta_2 + \delta_1{}^2}{a^2} + \ldots)\right]\right\} =$$

$$\frac{35g\beta q}{K}\left\{4a\left(\delta_o{}^3 + \frac{3\delta_o{}^2\delta_1}{a} + \frac{3\delta_o{}^2\delta_2 + 3\delta_o\delta_1{}^2}{a^2} + ..\right) + \left(\delta_o{}^4 + \frac{4\delta_o{}^3\delta_1}{a} + ..\right)\right\}$$

$$- 840\ a\nu\left(u_o + \frac{u_1}{a} + \frac{u_2}{a^2} + ..\right)$$

and

$$\left(u_o + \frac{u_1}{a} + \frac{u_2}{a^2} + \ldots\right)\left[7a\left(\delta_o{}^2 + \frac{2\delta_o\delta_1}{a} + \frac{2\delta_o\delta_2 + \delta_1{}^2}{a^2} + ..\right)\right. \qquad (3f)$$

$$\left. + 2\left(\delta_o{}^3 + \frac{3\delta_o{}^2\delta_1}{a} + \ldots.\right)\right] = \frac{\nu}{Pr}\, a\ 420\ x$$

After the indicated multiplications have been performed, the terms are arranged in descending powers of $\underline{a}$. The expression for conservation of momentum then takes the form

$$\left[\ 8\delta_o \frac{d}{dx}(\delta_o u_o{}^2) - \frac{140\ g\beta q}{K}\,\delta_o{}^3 + 840\nu u_o\right] a^1 + \left[\ \right] a^o \qquad (13)$$

$$+ \left[\ \right] a^{-1} + \ .\ .\ . = 0$$

Similarly, the expression for conservation of energy takes the form

$$\left[\ u_o\delta_o{}^2 - 60\,\frac{\nu}{Pr}\,x\ \right] a^1 + \left[\ \right] a^o + \left[\ \right] a^{-1} + \ .\ .\ . = 0 \qquad (14)$$

In order for above expressions to vanish identically for all values of radius, $\underline{a}$, each of the brackets must be identically zero. The pair of brackets multiplying a to the first power is a pair of simultaneous equations in δ_o and u_o which were solved by Sparrow [5] and were the result of a similar analysis as in Chapter II for a vertical flat plate with uniform heat flux. Consequently

$$u_o = \left\{ \left(\frac{\nu 60}{Pr}\right) \left[\frac{72(4 + 5\ Pr)K\nu^2}{Pr^2 g\beta q} \right]^{-\frac{2}{5}} \right\} x^{\frac{3}{5}} \tag{15}$$

and

$$\delta_o = \left[\frac{72(4 + 5\ Pr)K\nu^2}{Pr^2 g\beta q} \right]^{\frac{1}{5}} x^{\frac{1}{5}} \tag{16}$$

It will be convenient to define A_o and B_o such that

$$u_o = A_o\ x^{\frac{3}{5}} \tag{15a}$$

$$\delta_o = B_o\ x^{\frac{1}{5}} \tag{16a}$$

From manipulation of the coefficients of equations (15) and (16) it is found that

$$A_o = \left(\frac{\nu 60}{Pr}\right) \frac{1}{B_o^{\ 2}}$$

The pair of equations corresponding to a to the zero power are

$$2u_o\delta_o^{\,3} + 14u_o\delta_o\delta_1 + 7u_1\delta_o^{\,2} = 0 \tag{17}$$

and

$$\delta_o \frac{d}{dx}\left(3\delta_o^{\,2}u_o^{\,2} + 8\delta_1 u_o^{\,2} + 16\delta_o u_o u_1\right) \tag{18}$$

$$+ 8\delta_1 \frac{d}{dx}(\delta_o u_o^{\,2}) - \frac{35g\beta q}{K}\left\{12\delta_o^{\,2}\delta_1 + \delta_o^{\,4}\right\}$$

$$+ 840\ \nu u_1 = 0$$

Assuming the form of u_1, and δ_1, to be

$$u_1 = A_1\, x^{P_1} \tag{19}$$

$$\delta_1 = B_1\, x^{S_1} \tag{20}$$

Substitute equations (15a), (16a) and the assumed form of u_1 and δ_1 into equations (17) and (18). For equations (17) and (18) to be valid for all x

$$P_1 = \frac{4}{5}$$

and

$$S_1 = \frac{2}{5}$$

Using the relations for A_o and B_o in equation (17) yields

$$A_1 = -\left(\frac{\nu 60}{Pr}\right)\left[\frac{1}{14}\left(\frac{236 + 175\ Pr}{152 + 175\ Pr}\right)\right]\frac{1}{B_o} \tag{21}$$

Using the expressions for A_o, A_1, and B_o the expression for B_1 is found to be

$$B_1 = - \frac{372 + 525 \text{ Pr}}{28(158 + 175 \text{ Pr})} B_o^2 \tag{22}$$

For convenience defining

$$A_1 = A_o \alpha_1 B_o$$

and

$$B_1 = \gamma_1 B_o^2$$

where

$$\alpha_1 = - \frac{1}{14} \left(\frac{236 + 175 \text{ Pr}}{152 + 175 \text{ Pr}}\right) \tag{23}$$

$$\gamma_1 = - \frac{(372 + 525 \text{ Pr})}{28(152 + 175 \text{ Pr})} \tag{24}$$

The solving for A_1 and B_1 involved time-consuming algebra, which was only an indication of the laborious algebra required to obtain B_2. The equations corresponding to a to the minus one power are

$$14 u_o\delta_o\delta_2 + 7 u_o\delta_1^2 + 6 u_o\delta_o^2\delta_1 + 14 u_1\delta_o\delta_1 \tag{25}$$

$$+ 2 u_1\delta_o^3 + 7 u_2\delta_o^2 = 0$$

and

$$\delta_o \frac{d}{dx} \left\{ 8\, u_o^{\,2}\delta_2 + 6\, u_o^{\,2}\delta_o\delta_1 + 16\, u_o u_1 \delta_1 + 6\, u_o u_1 \delta_o^{\,2} \right. \tag{26}$$

$$\left. + 16\, u_o u_2 \delta_o + 8\, u_1^{\,2}\delta_o \right\} + \delta_2 \frac{d}{dx} \left\{ 8 u_o^{\,2}\delta_o \right\}$$

$$+ \delta_1 \frac{d}{dx} \left\{ 8\, u_o^{\,2}\delta_1 + 3\, u_o^{\,2}\delta_o^{\,2} + 16\, u_o u_1 \delta_o \right\}$$

$$= \frac{35 g\beta q}{K} \left\{ 4a(3\delta_o^{\,2}\delta_2 + 3\delta_o\delta_1^{\,2}) + 4\delta_o^{\,3}\delta_1 \right\}$$

$$- 840\, a\nu u_2$$

Continuing as before by assuming

$$u_2 = A_2\, x^{P_2}$$

and

$$\delta_2 = B_2\, x^{S_2}$$

For equations (25) and (26) to be valid for all x

$$P_2 = 1$$

and

$$S_2 = \frac{3}{5}$$

Using the known expressions for A_o, B_o, A_1 and B_1 the expression for B_2 was found after considerable algebraic manipulation to be

$$B_2 = \frac{B_o^{\,3}}{(328 + 350\ Pr)} \Big\{ \gamma_1(-\ 248\ \gamma_1 - \frac{710}{7} - 16\ \alpha_1) \tag{27}$$

$$+\ \alpha_1(\frac{90}{7} + 72\ \alpha_1)$$

$$-\ Pr\left[\gamma_1(280\ \gamma_1 + 130 + 140\ \alpha_1) + 20\ \alpha_1\ \right]\Big\}$$

For convenience defining

$$B_2 = \gamma_2\ B_o^{\,3}$$

where

$$\gamma_2 = \frac{1}{(328 + 35\ Pr)} \Big\{\gamma_1(-\ 248\ \gamma_1 - \frac{710}{7} - 16\ \alpha_1) \tag{28}$$

$$+\ \alpha_1(\frac{90}{7} + 72\ \alpha_1)$$

$$-\ Pr\left[\gamma_1(280\ \gamma_1 + 130 + 140\ \alpha_1) + 20\ \alpha_1\ \right]\Big\}$$

Now the first terms of the series for the boundary-layer thickness δ have been determined to be as follows

$$\delta = B_o\ x^{\frac{1}{5}} + B_1\ \frac{x^{\frac{2}{5}}}{a} + B_2\ \frac{x^{\frac{3}{5}}}{a^2} + \ .\ .\ . \tag{29}$$

It was disappointing at this point to note that it was not possible to recognize a recurrence relationship for B_n. As will be demonstrated, there seems to be little justification for undertaking the massive algebra that will be necessary to obtain B_3 and any of the succeeding terms.

CHAPTER FOUR

HEAT TRANSFER RESULTS

The definition of the local heat transfer coefficient is

$$h_x \equiv \frac{Q}{A\theta_w} = \frac{q}{\theta_w}$$

But by equation (6)

$$\theta_w = \frac{q\delta}{2K}$$

then

$$h_x = \frac{2K}{\delta}$$

and the local Nusselt number is

$$Nu_x = \frac{h_x x}{K} = \frac{2x}{\delta}$$

Using equation (29) Nu_x can be written as

$$Nu_{x_{CYL}} = \frac{2x}{B_o\, x^{\frac{1}{5}} + B_1 \frac{x^{\frac{2}{5}}}{a} + B_2 \frac{x^{\frac{3}{5}}}{a^2} + \ldots}$$

or after rearranging as

$$Nu_{x_{CYL}} = \frac{2x^{\frac{4}{5}}}{B_o} \quad \frac{1}{1 + \frac{B_1}{B_o}\frac{x^{\frac{1}{5}}}{a} + \frac{B_2}{B_o}\left(\frac{x^{\frac{1}{5}}}{a}\right)^2 + \ldots} \tag{29.1}$$

From Sparrow's solution [5] for a vertical flat plate with uniform heat flux

$$\delta_{fp} = \left[\frac{72(4 + 5\ Pr)K\nu^2}{Pr^2\ g\beta q}\right]^{\frac{1}{5}} x^{\frac{1}{5}} = B_o\ x^{\frac{1}{5}} \tag{30}$$

and

$$\theta_{w_{fp}} = \frac{q\delta}{2K} = \frac{qB_o}{2K}\ x^{\frac{1}{5}} \tag{31}$$

consequently

$$Nu_{x_{fp}} = \frac{h_{x_{fp}}\ x}{K} = \frac{2x^{\frac{4}{5}}}{B_o} \tag{32}$$

Equations (29.1) and (32) lead to the convenient relation

$$\frac{Nu_{x_{fp}}}{Nu_{x_{CYL}}} = 1 + \frac{B_1}{B_o}\frac{x^{\frac{1}{5}}}{a} + \frac{B_2}{B_o}\left(\frac{x^{\frac{1}{5}}}{a}\right)^2 + \ldots$$

Introducing the diameter, D, gives

$$\frac{Nu_{x_{fp}}}{Nu_{x_{CYL}}} = 1 + 2\,\frac{B_1\,x^{\frac{1}{5}}}{B_o\,D} + 4\,\frac{B_2\,x^{\frac{2}{5}}}{B_o\,D^2} + \ldots. \qquad (33)$$

and after rearranging

$$\frac{Nu_{x_{fp}}}{Nu_{x_{CYL}}} = 1 + \gamma_1\left\{2\,\frac{B_o\,x^{\frac{1}{5}}}{D}\right\} + \gamma_2\left\{2\,\frac{B_o\,x^{\frac{1}{5}}}{D}\right\}^2 + \ldots \qquad (34)$$

ϕ, which will be treated as the independent variable, is defined as

$$\phi = \left\{2\,\frac{B_o\,x^{\frac{1}{5}}}{D}\right\} = 2\left\{\left[\frac{72(4 + 5\,Pr)K\nu^2}{Pr^2\,g\beta q\,x^4}\right]^{\frac{1}{5}}\right\}\frac{x}{D} \qquad (35)$$

Sparrow [5] used a modified Grashof number, which is independent of the temperature difference, defined as

$$G_{r_x}^{*} = \frac{g\beta q\,x^4}{K\nu^2} = G_{r_x}\,Nu_x$$

Using experimental data for a vertical flat plate with uniform surface heat Sparrow and Gregg [6] showed that $G_{r_x}^{*}$ had a lower limit of 10^5, below which the boundary layer equations do not apply, and an upper limit of 10^{11}, above which the fluid flow would be in the transition region from laminar to turbulent flow. Since no experimental data are available for the cylindrical case, the same range will be assumed. Now the independent variable, ϕ, can be written as

$$\phi = 2 \left\{ \frac{72(4 + 5\,Pr)}{Pr^2\, G_{r_x}^{*}} \right\}^{\frac{1}{5}} \frac{x}{D} \tag{36}$$

Using equation (36) and the prescribed limits in $G_{r_x}^{*}$ the range of

$$\frac{x}{D}$$

for various values of Prandtl number has been tabulated in Table 1. Equations (34) written in terms of ϕ is

$$\frac{Nu_{x_{fp}}}{Nu_{x_{CYL}}} = 1 + \gamma_1\, \phi + \gamma_2\, \phi^2 + \ldots \tag{37}$$

The surface-temperature distribution can be found from the relation

$$\theta_w = \frac{q\delta}{2K}$$

Thus

$$\theta_{w_{CYL}} = \frac{q}{2K} \left\{ B_o\, x^{\frac{1}{5}} + 2B_1 \frac{x^{\frac{2}{5}}}{D} + 4B_2 \frac{x^{\frac{3}{5}}}{D^2} + \ldots \right. \tag{38}$$

which leads to

$$\theta_{w_{CYL}} = \frac{q}{2K} B_o x^{\frac{1}{5}} \left\{ 1 + 2 \frac{B_1 x^{\frac{1}{5}}}{B_o D} + 4 \frac{B_2 x^{\frac{2}{5}}}{B_o D^2} + \ldots \right\} \qquad (39)$$

But from equation (31)

$$\theta_{w_{fp}} = \frac{q B_o}{2K} x^{\frac{1}{5}}$$

thus

$$\frac{\theta_{w_{CYL}}}{\theta_{w_{fp}}} = 1 + \gamma_1 \phi + \gamma_2 \phi^2 + \ldots \qquad (40)$$

In a similar manner, the boundary-layer thickness at any point on the cylinder surface expressed as a ratio to the flat plate boundary-layer thickness is

$$\frac{\delta_{CYL}}{\delta_{fp}} = 1 + \gamma_1 \phi + \gamma_2 \phi^2 + \ldots \qquad (41)$$

Three ratios of constant surface heat flux conditions for the flat plate results (as developed by Sparrow [5]) to the cylindrical results have been developed. The flat plate results of Sparrow have been verified by a more exact analysis and by experimental data. The values of $Nu_{x_{fp}}$, $\theta_{w_{fp}}$ and δ_{fp} obtained by the Sparrow and Gregg solution [6] for the vertical flat plate with uniform surface heat flux have been verified; the most accurate method would be to use these values in equations (37), (40), and (41). This approach would demonstrate clearly the departure of the flat plate results due to the cylindrical geometry.

In accordance with the general practice of presenting heat transfer results, an expression for an average Nusselt number will be developed. The definition of the average Nusselt number is

$$\overline{Nu} \equiv \frac{\bar{h}L}{K}$$

In order to define an average heat transfer coefficient, it will be necessary to define a temperature difference. There is not an obvious characteristic temperature difference, but the most convenient one that would give a true representation would be the average temperature difference between $x = 0$ and $x = L$. Thus using the expression below to define $\overline{\theta}_w$

$$\overline{t_w - t_e} = \overline{\theta_w} = \frac{1}{L}\int_0^L \theta_w dx = \frac{1}{L}\int_0^L \frac{\delta q}{2K}\, dx$$

and by defining

$$\bar{h} \equiv \frac{Q}{A\ \overline{\theta_w}} = \frac{q}{\frac{1}{L}\int_0^L \frac{\delta q}{2K}\, dx}$$

leads to the relation

$$\overline{Nu} = \frac{\bar{h}L}{K} = \frac{\dfrac{q}{\dfrac{1}{L}\displaystyle\int_0^L \frac{\delta q}{2K}\,dx} \times L}{K}$$

which reduces to

$$\overline{Nu} = \frac{2L^2}{\displaystyle\int_0^L \delta dx}$$

Substituting in the series for δ and assuming that conditions are satisfied for termwise integration results in

$$\overline{Nu}_{CYL} = \frac{2L^2}{\frac{5}{6} B_o L^{\frac{6}{5}} + \frac{7}{5} B_1 \frac{L^{\frac{7}{5}}}{a} + \frac{8}{5}\frac{L^{\frac{8}{5}}}{a^2} + \ldots}$$

Rearranging and introducing the diameter, D,

$$\overline{Nu}_{CYL} = \frac{12}{5}\frac{L^{\frac{4}{5}}}{B_o} \qquad \frac{1}{1 + \frac{12}{7}\frac{B_1}{B_o}\frac{L^{\frac{1}{5}}}{D} + 3\frac{B_2 L^{\frac{2}{5}}}{B_o D^2} + \ldots} \tag{42}$$

From Sparrow's solution [5] for the flat plate

$$\theta_{w_{fp}} = \frac{q\ B_o}{2K}\ x^{\frac{1}{5}}$$

thus

$$\bar{h}_{fp} = \frac{Q}{A\ \frac{1}{L}\int_o^L \frac{q\ B_o}{2K}\ x^{\frac{1}{5}}\ dx}$$

$$\bar{h}_{fp} = \frac{12\ K}{5\ B_o L^{\frac{1}{5}}}$$

Consequently

$$\overline{Nu}_{fp} = \frac{12}{5}\frac{L^{\frac{4}{5}}}{B_o} \tag{43}$$

$\overline{Nu}_{fp}$ is the first factor on the right side of equation (42). Thus

$$\frac{\overline{Nu}_{fp}}{\overline{Nu}_{CYL}} = 1 + \frac{12\ B_1\ L^{\frac{1}{5}}}{7\ B_o\ D} + 3\ \frac{B_2\ L^{\frac{2}{5}}}{B_o\ D^2} + \ldots$$

and after substituting for B_1 and B_2

$$\frac{\overline{Nu}_{fp}}{\overline{Nu}_{CYL}} = 1 + \frac{12}{7}\ \gamma_1\ B_o\ \frac{L^{\frac{1}{5}}}{D} + 3\ \gamma_2\ B_o{}^2 \left(\frac{L^{\frac{1}{5}}}{D}\right)^2 + \ldots \tag{44}$$

Now by defining

$$\phi_L = \left\{2\ B_o\ \frac{L}{D}\right\}^{\frac{1}{5}}$$

equation (44) can be written as

$$\frac{\overline{Nu}_{fp}}{\overline{Nu}_{CYL}} = 1 + \frac{6}{7}\ \gamma_1\ \phi_L + \frac{3}{4}\ \gamma_2\ \phi_L^{\ 2} + \dots \tag{45}$$

Equation (45) compares the average Nusselt number for a cylinder of length L to that of a flat plate of the same length under the condition that q is the same for the flat plate and the cylinder.

All of the results that have been obtained thus far depend on the convergence of the series

$$1 + \gamma_1\ \phi + \gamma_2\ \phi^2 + \gamma_3\ \phi^3 + \dots$$

Unless a recurrent relation is developed for the remaining terms nothing definite can be concluded about the convergence of this series. The terms that have been found show all the properties necessary for convergence.

The range of γ_1 will be investigated. For the extreme values of Prandtl number the values of γ_1 are

$$\lim_{Pr \to 0} \gamma_1 = \lim_{Pr \to 0} \left\{- \frac{(372 + 525\ Pr)}{28(152 + 175\ Pr)}\right\}$$

$$= -\ 0.0873$$

and

$$\lim_{Pr \to \infty} \gamma_1 = - 0.107$$

Now taking the first derivative of γ_1 with respect to Pr in order to test for the existence of a relative maximum or minimum

$$\frac{d\ \gamma_1}{d\ Pr} = \frac{d}{d\ Pr} \left\{ - \frac{372 + 525\ Pr}{28(152 + 175\ Pr)} \right\}$$

$$= - \frac{1}{28} \left\{ \frac{(152 + 175\ Pr)\ 525 - (372 + 525\ Pr)\ 175}{(152 + 175\ Pr)^2} \right\}$$

$$= - \frac{175}{28} \left\{ \frac{84}{(152 + 175\ Pr)^2} \right\}$$

The first derivative is always less than zero and hence a relative maximum or relative minimum does not exist, and the maximum numerical value of γ_1 is for Prandtl number approaching infinity.

In the same manner for α_1

$$\lim_{Pr \to 0} \alpha_1 = - 0.111$$

and

$$\lim_{Pr \to \infty} \alpha_1 = - 0.0715$$

The first derivative of α_1 with respect to Prandtl number is always greater than zero, hence there are no maximum or minimum values between the extreme points. The maximum numerical value of α_1 is for Prandtl number near zero.

The range of γ_2 can be demonstrated by various values of Prandtl number but no proof that it is monotonic will be presented since the work involved in obtaining the first derivative of γ_2 with respect to Prandtl number is prohibitive. Thus the corresponding values of γ_2 are

Pr	γ_2
0	0.0121
0.1	0.0205
1	0.0258
10	0.0308
100	0.0316
∞	0.0318

Thus it appears that the maximum value of γ_2 is for the Prandtl number approaching infinity.

It was pointed out at the end of the last chapter that it was not practical to solve for γ_3. But as will be demonstrated in the next chapter, it is possible to make simplifying assumptions for the case of large Prandtl number*. A result of solving equation (50) (as presented in the next chapter) in the same manner as equations (2e) and (3e) is a value of

$$\gamma_3 = -0.0121$$

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*For the purposes of this analysis the term "large Prandtl number" will refer to any Prandtl number greater than 100.

The series shows the encouraging trend of having alternating signs and the coefficients of ϕ decrease as the powers of ϕ increase. If these trends continue and if the restriction that

$$\phi < 1$$

is imposed, convergence of the series is assured.

A closed form solution for large values of Prandtl number, that does not depend on the convergence of a series, will be presented in Chapter V. Thus it will be possible to compare on Figure 2* the foregoing series solution with the closed form solution for large values of the Prandtl number.

An important characteristic of a convergent alternating series is that the error after n terms is numerically less than the (n + 1)st term. By postulating an alternating converging series and assuming

$$\gamma_{2_{max}} = 0.0318$$

the error due to truncation of the series for ϕ near unity is

$$\text{error} < \gamma_{2_{max}} \phi = 0.0318$$

From Figure 2 the worst case for convergence, i.e., that of ϕ near one, the sum of the series for any Prandtl number is always greater than 0.89.

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*It should be pointed out that closeness of the curves for the range of the Prandtl number gives a false indication of the effect of the Prandtl number since the ordinate, ϕ, also is affected by Prandtl number.

Thus the error expressed as a percentage is

$$\text{error} < \frac{0.0318}{0.89} \times 100 = 3.6\%$$

The 3.6 per cent error is actually a conservative estimate for the error of using only two terms of the series. In the foregoing solution, three terms of the series were used and the error would be less.

It is interesting to note that the maximum values of γ_1 and γ_2 occurred for large values of Prandtl number. Thus the smaller the Prandtl number the smaller the error for the same value of ϕ. Also, if it were possible to show that this trend continued, then the value of γ_3 obtained for the case of large Prandtl number would indicate that the error due to truncation for ϕ near unity would be less than 0.0121 or

$$\text{error} < \frac{0.0121}{0.89} \times 100 \approx 1.4\%$$

A simplified expression for the local Nusselt number can be arrived at by using a method similar to that employed by LeFevre and Ede [3] for the case of a vertical cylinder with uniform surface temperature. LeFevre and Ede were able to realize a considerable simplification by choosing to use only two terms of the series for the boundary-layer thickness, δ. They felt that the assumptions they had made did not justify using any more than two terms.

Following LeFevre and Ede by using only two terms of series for δ

$$Nu_x = \frac{2x}{\delta} \approx \frac{2x}{B_o x^{\frac{1}{5}} + B_1} \frac{x^{\frac{1}{5}}}{a}$$

and after rearranging

$$Nu_x = \frac{2x}{B_o\, x^{\frac{1}{5}} \left(1 + \frac{B_1\, x^{\frac{1}{5}}}{B_o\, a}\right)}$$

Expanding the denominator in a series and assuming that

$$\frac{B_1\, x^{\frac{1}{5}}}{B_o\, a} = \gamma_1\, \phi$$

is small in comparison with unity. In the discussion of the convergence of the series

$$1 + \gamma_1 \phi + \gamma_2 \phi^2 + \ldots$$

γ_1 was shown to be no larger than 0.107 and ϕ was restricted to values less than one. Thus

$$\gamma_1 \phi = \frac{B_1\, x^{\frac{1}{5}}}{B_o\, a} < 0.107$$

The local Nusselt number is approximated by

$$Nu_x = \frac{2x^{\frac{4}{5}}}{B_o} - \frac{2\, B_1}{B_o^{\,2}}\, \frac{x}{a} \qquad (46)$$

if only two terms of the series expansion are used. Substituting for B_o and B_1 and introducing diameter, D, results in

$$Nu_{x_{CYL}} = 2\left\{\frac{Pr^2\, G_{r_x}^*}{72(4 + 5\ Pr)}\right\}^{\frac{1}{5}} + \frac{1}{7}\left\{\frac{372 + 525\ Pr}{(152 + 175\ Pr)}\right\}\frac{x}{D} \tag{47}$$

where $G_{r_x}^*$ is the modified Grashof number defined by equation (35).

Rearranging into a form that can be more readily compared to the first expression for Nu_x obtained, equation (37), gives

$$\frac{\dfrac{Nu_{x_{fp}}}{\left[Pr\ G_{r_x}^*\right]^{\frac{1}{5}}}}{\dfrac{Nu_{x_{CYL}}}{\left[Pr\ G_{r_x}^*\right]^{\frac{1}{5}}}} = \frac{2\left\{\dfrac{Pr}{72(4 + 5\ Pr)}\right\}^{\frac{1}{5}}}{2\left\{\dfrac{Pr}{72(4 + 5\ Pr)}\right\}^{\frac{1}{5}} + \dfrac{1}{7}\left\{\dfrac{372 + 525\ Pr}{152 + 175\ Pr}\right\}\left[Pr\ G_{r_x}^*\right]^{-\frac{1}{5}}\dfrac{x}{D}}$$

or

$$\frac{Nu_{x_{fp}}}{Nu_{x_{CYL}}} = \frac{2\left\{\dfrac{Pr}{72(4 + 5\ Pr)}\right\}^{\frac{1}{5}}}{2\left\{\dfrac{Pr}{72(4 + 5\ Pr)}\right\}^{\frac{1}{5}} + \dfrac{1}{7}\left\{\dfrac{372 + 525\ Pr}{152 + 175\ Pr}\right\}\left\{Pr\ G_{r_x}^*\right\}^{-\frac{1}{5}}\dfrac{x}{D}} \tag{48}$$

A further rearrangement of equation (48) gives the following form that permits a direct comparison to the first method

$$\frac{Nu_{x_{fp}}}{Nu_{x_{CYL}}} = \frac{1}{1 + \left\{\frac{(372 + 525\ Pr)}{28(152 + 175\ Pr)}\right\}\left(\frac{2\ B_o x^{\frac{1}{5}}}{D}\right)} \tag{49}$$

and by equations (24) and (35)

$$\frac{Nu_{x_{fp}}}{Nu_{x_{CYL}}} = \frac{1}{1 - \gamma_1 \phi}$$

The agreement between equation (49) and the more exact equation (37) can be seen from Figure 3.

CHAPTER V

CLOSED FORM RESULTS FOR LARGE VALUES OF THE PRANDTL NUMBER

In many cases the velocities in free convection are so small that the resulting inertia forces can be neglected relative to the viscous forces. Thus for certain values of the Prandtl number, the left side of equation (2e) could be neglected. Eckert and Drake [11] stated that for air (Pr = 0.72) fair agreement was obtained by using this assumption. For larger values of the Prandtl number, the agreement would be better. For the limiting case of large values of the Prandtl number, the left side of equation (2e) goes to zero. Using this assumption, equation (2e) becomes

$$\frac{35 g\beta q}{K}\left\{ (4a + \delta)\delta^3 \right\} - 840\, a\nu u^* = 0$$

Substituting for u^* as expressed by equation (3e) results in the following expression in δ and x

$$\frac{35 g\beta q}{K}\left\{ (4a + \delta)\delta^3 \right\} = 840\, a\nu \left\{ \frac{\nu}{Pr} \left[\frac{a\ 420\ x}{7a\delta^2 + 2\delta^3} \right] \right\} \qquad (50)$$

By manipulation of terms, equation (50) becomes

$$2\left[\left(\frac{\delta}{D}\right)^5 + \frac{15}{14}\left(\frac{\delta}{D}\right)^6 + \frac{2}{7}\left(\frac{\delta}{D}\right)^7 \right]^{\frac{1}{5}} = 2\left\{ \frac{360}{Pr\ G_{r_x}^{\ *}} \right\}^{\frac{1}{5}} \frac{x}{D} \qquad (51)$$

From equation (36)

$$\phi = 2 \left\{ \frac{72(4 + 5\ Pr)}{Pr^2\ G_{r_x}^*} \right\}^{\frac{1}{5}} \frac{x}{D}$$

and if

$$Pr\ G_{r_x}^*$$

in the denominator is considered separately, then taking the limit for large values of the Prandtl number results in

$$\lim_{Pr \to \infty} \phi = \frac{2}{(Pr\ G_{r_x}^*)} \frac{x}{D} \left[\lim_{Pr \to \infty} \left\{ \frac{72(4 + 5\ Pr)}{Pr} \right\}^{\frac{1}{5}} \right]$$

$$= 2 \left\{ \frac{360}{Pr\ G_{r_x}^*} \right\}^{\frac{1}{5}} \frac{x}{D} = \frac{6.490}{(Pr\ G_{r_x}^*)^{\frac{1}{5}}} \frac{x}{D}$$

It is interesting to note that for Pr = 10

$$\phi = \frac{6.60}{(Pr\ G_{r_x}^*)^{\frac{1}{5}}} \frac{x}{D} \qquad (52)$$

and for Pr = 100

$$\phi = \frac{6.496 \qquad x}{(\Pr G_{r_x}^{*})^{\frac{1}{5}} \quad D}$$

Thus for large values of the Prandtl number, the right side of equation (51) approaches the value of ϕ. Consequently, equation (51) can be written as

$$2 \left[(\tfrac{\delta}{D})^{5} + \tfrac{15}{14}(\tfrac{\delta}{D})^{6} + \tfrac{2}{7}(\tfrac{\delta}{D})^{7} \right] = \phi$$

for large values of the Prandtl number. As can be seen from equation (52), letting ϕ equal the right side of equation (51) is a good assumption for values of the Prandtl number as small as ten.

For a particular value of $\frac{\delta}{D}$ it is possible to find a corresponding value of ϕ. Or for a particular value of

$$\Pr G_{r}^{*}$$

$\frac{x}{D}$ is expressed as a function of $\frac{\delta}{D}$.

Recalling equation (30)

$$\delta_{fp} = \left\{ \frac{72(4 + 5 \Pr)K\nu^{2}}{\Pr^{2}\, g\beta q} \right\}^{\frac{1}{5}} x^{\frac{1}{5}}$$

and multiplying both sides by $\frac{2}{D}$ results in

$$2\left(\frac{\delta_{fp}}{D}\right) = 2\left\{\frac{72(4 + 5\ Pr)K\nu^2}{Pr^2\ g\beta q\ x^4}\right\}^{\frac{1}{5}} \frac{x}{D}$$

$$= \phi$$

Thus for a particular value of ϕ a corresponding value of $\frac{\delta_{fp}}{D}$ can be found.

Now from equation (32)

$$\frac{Nu_{x_{fp}}}{Nu_{x_{CYL}}} = \frac{\frac{2x}{\delta_{fp}}}{\frac{2x}{\delta}} = \frac{\frac{\delta}{D}}{\frac{\delta_{fp}}{D}} \tag{53}$$

$$= \frac{\frac{\delta}{D}}{\frac{\phi}{2}}$$

Equation (53) is a closed form expression and unlike equation (37) does not depend on the convergence of a series. Equations (53) and (37) are plotted on Figure 2 for the case of large Prandtl number. The agreement between equations (53) and (37) is shown on Figure 2. This agreement demonstrates the degree of accuracy of the truncated series for large values of Prandtl number and presents a further argument that the series of equation (37) converges.

CHAPTER VI

CONCLUSION AND RECOMMENDATIONS

An analysis of the laminar boundary layer equations, as they applied to free convection on a vertical cylinder with uniform surface heat flux, has been presented for

$$10^5 < G_{r_x}^* < 10^{11}$$

and

$$0.01 < Pr < 1000$$

The heat transfer results obtained by the solution of these equations are presented as curves in Figures 2, 3, and 4.

At present a systematic set of experimental data and results has not been published for laminar free convection on a vertical cylinder with uniform surface heat flux. Such experimental data would be helpful in establishing the validity and range of the analytical solution.

Further mathematical attacks on the differential equations would be desirable in order to find a more exact solution. A numerical solution might be a possible approach, if a similarity transformation could be found.

The case where the surface heat flux was not uniform, but varied as some function of height, should be investigated in a manner similar to Sparrow and Gregg [12] for a flat plate. Yang [7] showed that it would be possible to obtain a similarity solution for a vertical cylinder with surface heat flux that varies linearly with x, but unfortunately this linear variation cannot be used for the case of uniform surface heat flux.

APPENDIX A

APPENDIX A

INTRODUCTION OF DIMENSIONLESS VARIABLES

The integrated form of the expression for conservation of momentum is

$$\frac{d}{dx}\int_{a}^{a+\delta} u^2 r dr = g\beta \int_{a}^{a+\delta} \theta(x,r) r d - a\nu \left.\frac{du}{dr}\right)_{r=a}$$

The above expression will be made dimensionless by introduction of the dimensionless variables

$$\tilde{r} = \frac{r}{a}$$

$$\tilde{x} = \frac{x}{a}$$

$$\tilde{v} = \frac{av}{\nu}$$

$$\tilde{u} = \frac{au}{\nu}$$

$$\tilde{\delta} = \frac{\delta}{a}$$

Thus

$$\frac{1}{a}\frac{d}{d\tilde{x}}\int_{1}^{1+\tilde{\delta}} \left(\frac{\nu\tilde{u}}{a}\right)^2 a\tilde{r}\, ad\tilde{r}$$

$$= g\beta\int_{1}^{1+\tilde{\delta}} \theta(x,r)a\tilde{r}\, ad\tilde{r} - a\nu\,\frac{\nu}{a^2}\,\frac{\partial\tilde{u}}{\partial\tilde{r}}\Big)_{\tilde{r}=1}$$

Introducing

$$y = r - a$$

or in nondimensional form

$$\tilde{y} = \tilde{r} - 1$$

results in a nondimensional form for the conservation of momentum expression

$$\frac{\nu^2}{a}\frac{d}{d\tilde{x}}\int_{o}^{\tilde{\delta}}\tilde{u}^2(\tilde{y} + 1)d\tilde{y}$$

$$= g\beta a^2\int_{o}^{\tilde{\delta}} \theta(x,r)(1 + \tilde{y})d\tilde{y} - \frac{\nu^2}{a}\,\frac{\partial\tilde{u}}{\partial\tilde{y}}\Big)_{\tilde{y}=0}$$

The integrated form of the expression for conservation of energy is

$$\frac{d}{dx}\int_{a}^{a+\delta}\theta(x,r)urdr = -a\alpha\left.\frac{\partial\theta}{\partial r}\right)_{r=a}$$

The above expression will be made dimensionless by introduction of the dimensionless variables. Thus

$$\frac{1}{a}\frac{d}{d\tilde{x}}\int_{1}^{1+\tilde{\delta}}\theta(x,r)\,\frac{\nu}{a}\,\tilde{u}\;a\tilde{r}\;ad\tilde{r}$$

$$= -a\alpha\left.\frac{1}{a}\frac{\partial\theta(x,r)}{\partial\tilde{r}}\right)_{\tilde{r}=1}$$

Introducing

$$\tilde{y} = \tilde{r} - 1$$

results in a nondimensional form for the conservation of energy expression

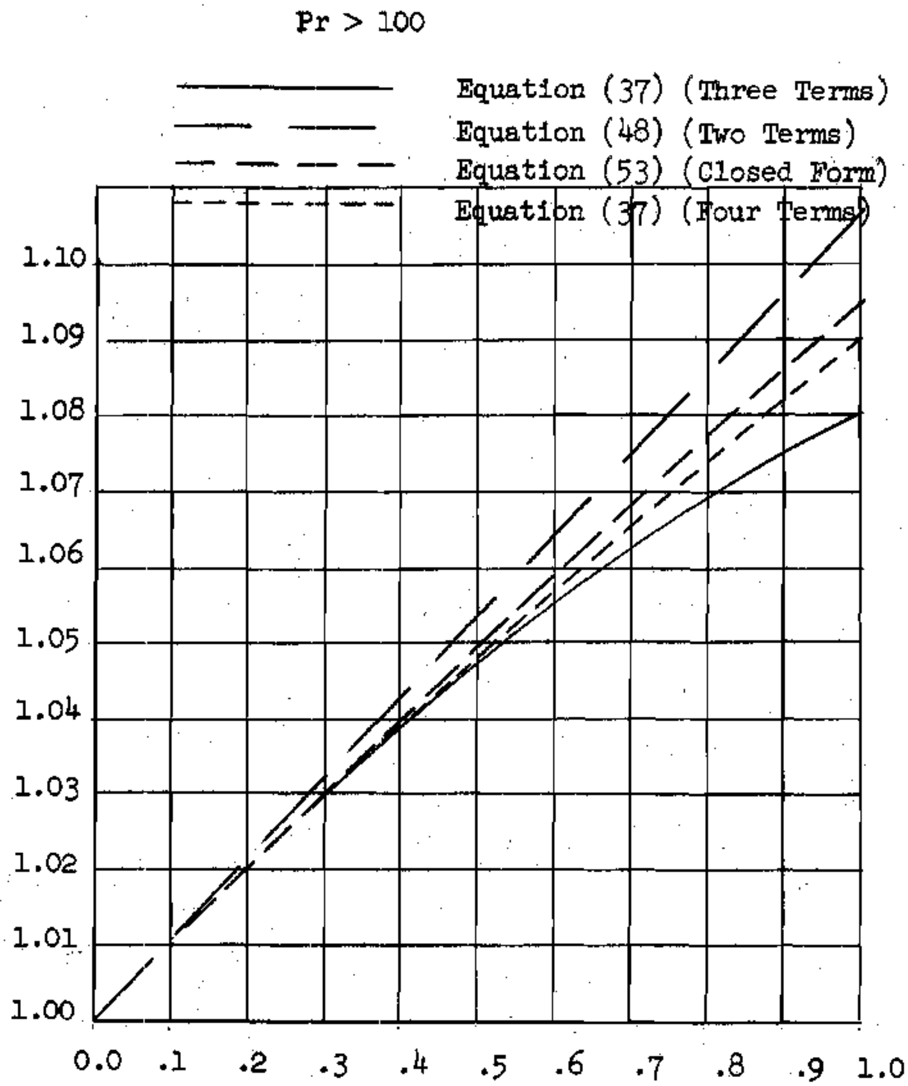
$$\frac{d}{d\tilde{x}}\int_{o}^{\tilde{\delta}}\nu\;\theta(x,r)(1+\tilde{y})d\tilde{y} = -\alpha\left.\frac{\partial\theta(x,r)}{\partial\tilde{y}}\right)_{\tilde{y}=0}$$

APPENDIX B

Table 1

Range of $\frac{x}{D}$ for Various Values of the Prandtl Number

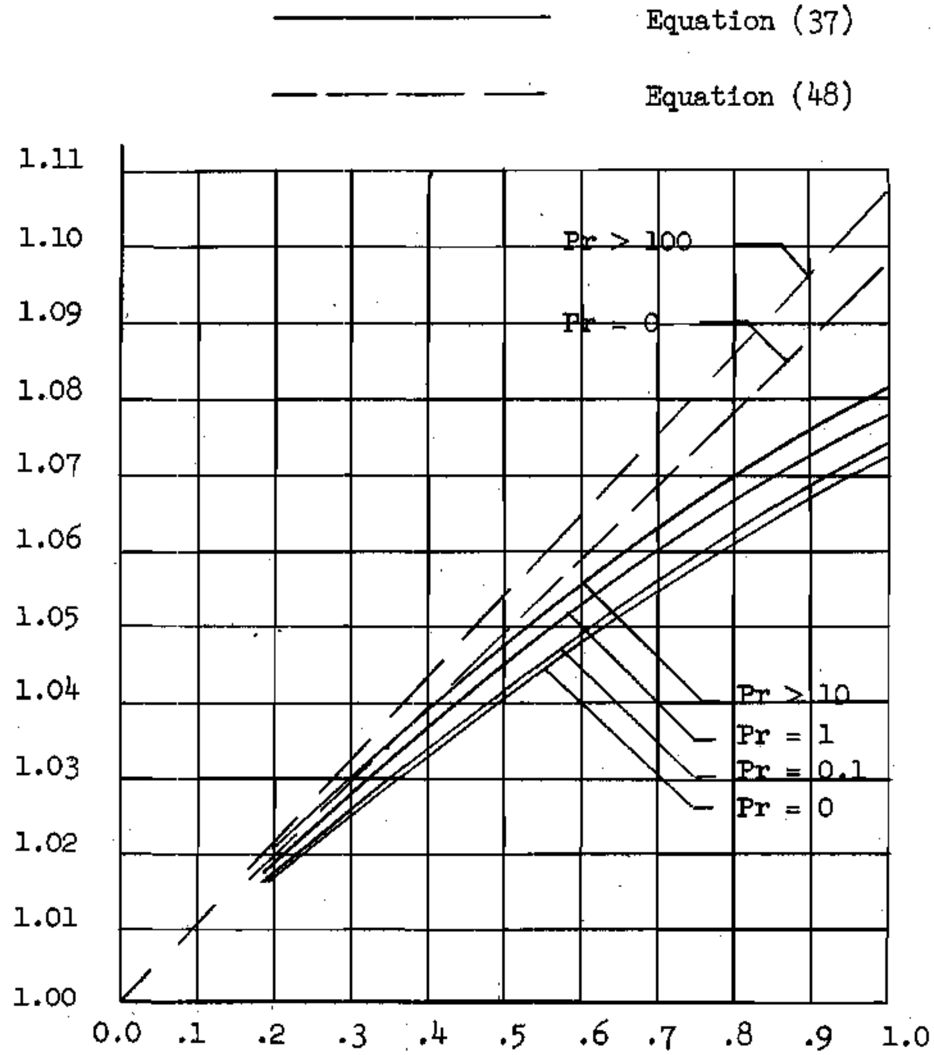
Pr	Range of $\frac{x}{D}$ for $G_{r_x}^{*} = 10^5$	Range of $\frac{x}{D}$ for $G_{r_x}^{*} = 10^{11}$
100	$0 \leq \frac{x}{D} < 3.9$	$0 \leq \frac{x}{D} < 62$
10	$0 \leq \frac{x}{D} < 2.4$	$0 \leq \frac{x}{D} < 38$
1	$0 \leq \frac{x}{D} < 1.4$	$0 \leq \frac{x}{D} < 22$
0.1	$0 \leq \frac{x}{D} < .7$	$0 \leq \frac{x}{D} < 11$



$$\frac{Nu_{x_{CYL}}}{Nu_{x_{fp}}}$$

$$\phi = 2 \left\{ \frac{72(4 + 5\,Pr)}{Pr^2\, G_{r_x}^{*}} \right\}^{\frac{1}{5}} \frac{x}{D}$$

Figure 2. Comparison of the Four Expressions of the Heat Transfer Results. Curves are for Large Values of Prandtl Number.

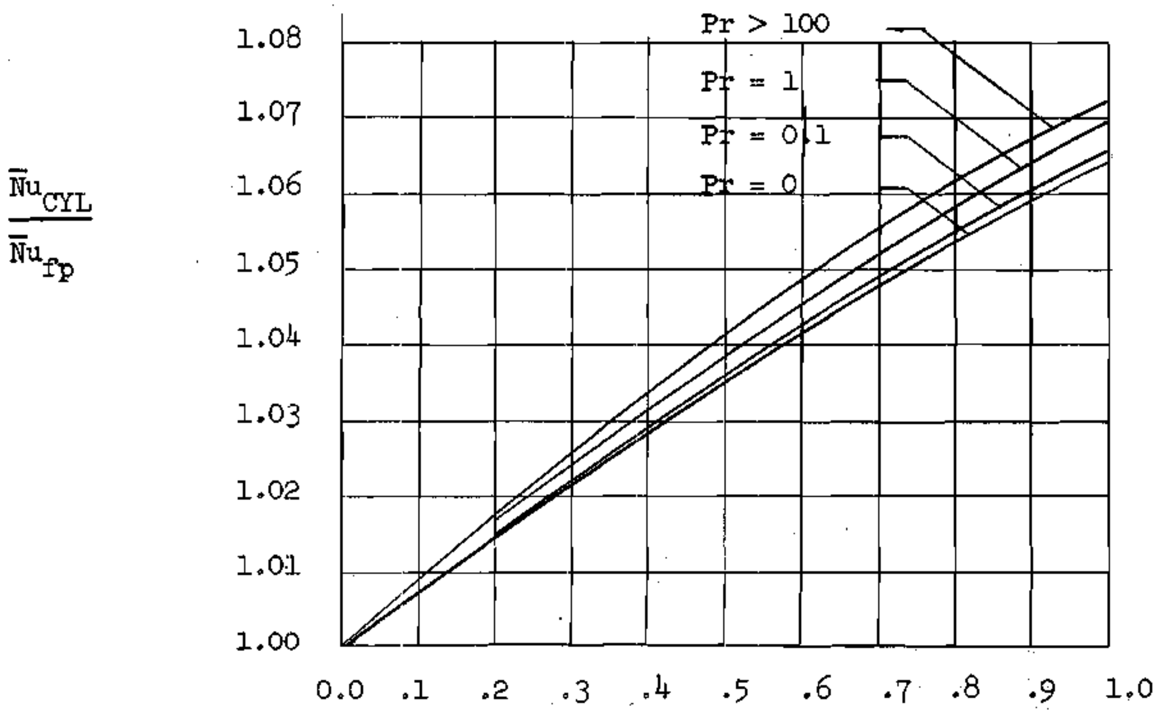


$$\frac{Nu_{CYL}}{Nu_{x_{fp}}} \text{ or } \frac{\delta_{fp}}{\delta_{CYL}} \text{ or } \frac{\theta_{fp}}{\theta_{CYL}}$$

$$\phi = 2\left\{\frac{72(4 + 5\ Pr)}{Pr^2\ G_{r_x}^*}\right\}^{\frac{1}{5}} \frac{x}{D}$$

Figure 3. Comparison of the Heat Transfer Results as Expressed by Equations (37) and (48). Curves are for Prandtl Number of 0, 0.1, 1, 10 and Pr > 100.

Equation (45)



$$\phi_L = 2 \left\{ \frac{72(4 + 5\ Pr)}{Pr^2\ G_{r_L}^{*}} \right\}^{\frac{1}{5}} \frac{L}{D}$$

Figure 4. Comparison of Average Nusselt Number Ratio. Curves are for Prandtl Numbers of 0, 0.1, 1 and Pr > 10.

BIBLIOGRAPHY

1. Sparrow, E. M., and Gregg, J. C., "Laminar Free Convection Heat Transfer from the Outer Surface of a Vertical Circular Cylinder," Transactions of The American Society of Mechanical Engineers, vol. 78, No. 1956, p. 1823-1829.

2. LeFevre, E. J., and Ede, A. J., "Laminar Free Convection from the Outer Surface of a Vertical Cylinder," Ninth International Congress of Applied Mechanics, 1956 Proceedings, vol. 4, p. 175-183.

3. Hama, R. H., and Recesso, J. V., "The Axisymmetric Free-Convective Temperature Field Along a Vertical Thin Cylinder," University of Maryland Institute of Fluid Dynamics and Applied Mathematics, TN BN-116, Jan. 1958.

4. Millsaps, K., and Pohlhausen, K., "The Laminar Free Convective Heat Transfer from the Outer Surface of a Vertical Circular Cylinder," Journal of the Aeronautical Sciences, June 1958, p. 357-360.

5. Sparrow, E. M., "Laminar Free Convection on a Vertical Plate with Prescribed Nonuniform Wall Temperature," National Advisory Committee for Aeronautics Technical Note 3508, July 1958.

6. Sparrow, E. M., and Gregg, J. L., "Laminar Free Convection from a Vertical Plate with Uniform Surface Heat Flux," Transactions of The American Society of Mechanical Engineers, vol. 78, 1956, p. 435-440.

7. Kwang-Tzu Yang, "Possible Similarity Solutions for Laminar Free Convection on Vertical Plates and Cylinders," Journal of Applied Mechanics, Paper No. 59-A-86, Feb. 1959.

8. Eckert, E. M. G., and Drake, R. M., Heat and Mass Transfer, McGraw Hill Book Company, Inc., 1959, p. 311-328.

9. Ibid., p. 316.

10. Ibid., p. 312.

11. Ibid., p. 236 and 317.

12. Sparrow, E. M., and Gregg, J. L., "Similar Solutions for Free Convection from a Nonisothermal Vertical Plate," Transactions of the American Society of Mechanical Engineers, vol. 80, 1958, p. 379-386.